CSE 291 / Math 277A - Quantum Complexity Theory (Fall 2025)
Homework 2
Due Friday, October 17, 11:59pm

Instructions: Note: It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. You may work with 1-2 other collaborators, but you must write the solutions separately and clearly mark the names of all people you worked with on each problem.

Problems:

1. **Trace distance and the max distinguishing probability of quantum states**

   How do we determine how close two n-qubit quantum states with density matrices ρ and σ are to each other? One popular metric is called the *trace distance*:

   $$\|\rho - \sigma\|_{\mathrm{tr}} = \frac{1}{2} \max_U \mathrm{tr} \left| U\rho U^\dagger - U\sigma U^\dagger \right|,$$

   where the maximum is over all unitary matrices U, and $\mathrm{tr}\,|\cdot|$ is the sum of the absolute values of the elements along the diagonal. (Note that because the unitary matrices form a compact group, the maximum in the definition is equivalent to a supremum.)

   (a) The trace distance is particularly useful because it is related to the maximum probability of distinguishing two quantum states: suppose you are given state ρ with probability 1/2 and state σ with probability 1/2. Prove that the highest probability with which you can determine the state given to you is

   $$\frac{1 + \|\rho - \sigma\|_{\mathrm{tr}}}{2}.$$

   You may assume that the strategy consists of applying some unitary matrix U followed by a computational basis measurement where the measurement outcomes are partitioned into two sets. If the measurement falls in one set, you conclude that the state you were given was ρ; otherwise, it falls in the other set, and you conclude that the state was σ. (Hint: write out the probability of success for any strategy, and then take the maximum.)

   (b) Show that the trace distance satisfies the triangle inequality: for all density matrices ξ, $\|\rho - \sigma\|_{\mathrm{tr}} \leq \|\rho - \xi\|_{\mathrm{tr}} + \|\sigma - \xi\|_{\mathrm{tr}}$.

2. **Quantum errors only accumulate linearly**

   Suppose we wish to apply the n-qubit quantum operation $U = U_t U_{t-1} \cdots U_1$, where unitary U is the product of t individual gates. Unfortunately, for each gate, there is some noise in our implementation so that when we try to apply U_i, we actually apply V_i. Thankfully, for any state ρ and any gate U_i, we have the guarantee that $\|V_i \rho V_i^\dagger - U_i \rho U_i^\dagger\|_{\mathrm{tr}} \leq \epsilon$. Prove that these errors only add up linearly as we apply the entire sequence of gates that computes U—that is, setting $V = V_t V_{t-1} \cdots V_1$, show that $\|V\rho V^\dagger - U\rho U^\dagger\|_{\mathrm{tr}} \leq t\epsilon$.

3. **Quantum computation does not require complex numbers**

   Suppose we have an n-qubit circuit constructed from a sequence of 1- and 2-qubit gates $g_1, \ldots, g_m$. Show that there is another $(n+1)$-qubit circuit constructed from gates $g'_1, \ldots, g'_m$ such that

   - Each gate g'_i can be efficiently constructed from g_i and only has *real* entries.
   - The probability distribution resulting from measuring the first qubit is the same for each circuit.

   (Hint: each gate g'_i may act on more qubits than g_i.)

4. **Project precursor problems**

   The purpose of this problem is to practice generating research ideas in quantum complexity theory. Concretely, your goal is to write down (at least) 2 research questions that you don't know the answer to. You should write these questions with the intention that one of them may become the basis for your final project in this class (you will repeat this exercise on future homework).

   For each question, you should pick some topic/theorem/area that we've learned about in class and propose some way to extend it. To reiterate—the purpose is to practice coming up with interesting *questions*, not necessarily answers. You also shouldn't yet worry about whether or not your question has been already answered somewhere in the quantum literature.

   To give you some example of this approach, consider the No-Cloning Theorem that we learned in class. That theorem states that there is no 2-qubit unitary U such that

   $$U|\psi\rangle \otimes |0\rangle = |\psi\rangle \otimes |\psi\rangle$$

   for all single-qubit states $|\psi\rangle$. Notice that this theorem requires an *exact* copy. What if we changed the problem to allow for an approximation? Here's an attempt at formalizing such a question:

   Research Question: What is the greatest positive value $\delta \in \mathbb{R}$ such that there exists a unitary U such that

   $$U|\psi\rangle \otimes |0\rangle = |\psi\rangle \otimes |\varphi\rangle$$

   where $|\langle\psi|\varphi\rangle|^2 \geq \delta$ for all single-qubit states $|\psi\rangle$?