CSE 291 / Math 277A - Quantum Complexity Theory (Fall 2025)
Homework 1
Due Friday, October 3, 11:59pm

Instructions: Note: It is highly recommended (though not required) that you type your answers. It is your responsibility to make any handwriting clear and legible for grading. You may work with 1-2 other collaborators, but you must write the solutions separately and clearly mark the names of all people you worked with on each problem.

Much of the homework in this course requires a background in linear algebra. Those of you who would like a refresher on some of these concepts are encouraged to look at Appendix A of Ronald de Wolf's quantum computing lecture notes.

Problems:

1. **Stochastic matrices and unitary matrices**

   (a) Prove $A \in \mathbb{R}^{n\times n}$ maps every non-negative vector $v \in \mathbb{R}^n$ to another non-negative vector with the same ℓ_1-norm if and only if A is *stochastic*—i.e., A is a nonnegative matrix such that the sum along every column is 1.

   (b) Prove $U \in \mathbb{C}^{n\times n}$ maps every vector $v \in \mathbb{C}$ to another vector with the same ℓ_2-norm if and only if U is *unitary*—i.e., $U^\dagger U = I$.

2. **Density Matrices and Distinguishing Ensembles of Quantum States**

   Not all quantum systems on n qubits can be described by a pure state $|\psi\rangle \in \mathbb{C}^{2^n}$. General quantum systems are fully described by statistical mixtures of quantum states—that is, an ensemble of pure states $\{|\psi_i\rangle\}_i$ each of which is prepared with probability $p_i \in [0,1]$. The *density matrix* corresponding to this ensemble is

   $$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \in \mathbb{C}^{2^n \times 2^n}$$

   where $\sum_i p_i = 1$.

   (a) Show—as a consequence of applying the measurement Born rule to each pure state in the ensemble—that when you measure a state with density matrix ρ, you get outcome i with probability equal to the (i,i) entry of ρ. (The "Born rule" refers to the quantum mechanical principle that measurement of the state $|\psi\rangle$ results in outcome i with probability equal to the square of the magnitude of the amplitude on $|i\rangle$.)

   (b) Show that when you apply unitary U to a system with density matrix ρ that the density matrix of the new system is $U\rho U^\dagger$.

   (c) Using parts (a) and (b), conclude that if two ensembles of states have the same density matrix that there is no experiment that can distinguish them. You can assume that any experiment consists of applying a unitary and then measuring in the computational basis.

(d) Prove that there are multiple ensembles that give rise to the same density matrix.

3. **Quantum Mechanics is Uniquely Nontrivial**

In this problem we will show that the ℓ_1 and ℓ_2 norms are special in that they are the only ℓ_p-norms that lead to interesting linear dynamics. While this will be true in general, we will restrict our attention to real numbers and even p for this problem. The ℓ_p-norm of a real vector $v \in \mathbb{R}^n$ is defined as

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p},$$

and matrix A preserves the ℓ_p-norm if for all $v \in \mathbb{R}^n$

$$\|Av\|_p = \|v\|_p.$$

It turns out that for every $p \geq 3$, a matrix $A \in \mathbb{R}^{n \times n}$ can only preserve the ℓ_p-norm if it is the product of a permutation matrix and a diagonal matrix. Prove this fact for every *even* $p \geq 4$. Hint: Start with a fixed norm like the 4-norm, and then generalize.