

Instructions: There are two parts to this homework:

- Concept check (Problem 1): Complete individually on Gradescope.
- Written Homework (Problem 2–3): You may work individually or in a team of up to 3 people. If working in a group, make only one submission per group on Gradescope: one person uploads the submission and then adds the other group members to the submission by selecting their names in the “Add Group Members” dialog box. All problems will be graded for completeness. Completeness means that it looks like there was a good-faith effort to solve the problem. A LaTeX template for the homework is provided on Canvas.

For many of the problems below, I ask you to “prove” some fact. In general, there is no specific structure of a proof that I am looking for in this class. Most of the proofs of the identities below can be shown by just computing two quantities and showing that they are evidently the same.

Problems:

1. Concept check

Complete the assignment “Homework 1 - Concept Check” on Gradescope. It consists of a few multiple choice questions that test some of the basic concepts we’ve learned in the class. A detailed explanation will appear if you’ve gotten the answer correct. The score you get on this part of the assignment *will* contribute to your homework grade, but you have unlimited attempts to get the right answer.

2. Complex numbers fundamentals

Quantum mechanical systems are described by vectors of complex numbers. Therefore, it is important to review some key properties of complex numbers. We will use the symbol $\mathbb{R}$ to denote the set of all real numbers and the symbol $\mathbb{C}$ to denote the set of all complex numbers. That is,

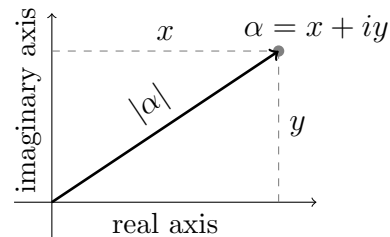
$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } i^2 = -1\}$$

For a complex number $\alpha = x + iy$, we say that x is the *real part* of α and y is the *imaginary part* of α .

- (a) **Complex conjugate:** for every complex number $\alpha = x + iy$, we define its *complex conjugate* as $\alpha^* := x - iy$. As a note of warning, we will mostly write the complex conjugate as α^* , but it is also common to see it written as $\bar{\alpha}$. Let $\alpha, \beta \in \mathbb{C}$ be any two complex numbers. Prove the following two identities:

- $(\alpha\beta)^* = \alpha^*\beta^*$
- $(\alpha + \beta)^* = \alpha^* + \beta^*$

- (b) **Magnitude:** As shown on the right, we can think of a complex number $\alpha = x + iy$ as a vector in 2-dimensional space, where the real and imaginary parts are the two components. The *absolute value* or *magnitude* of α , which we denote by $|\alpha|$, is just the length of this vector. That is, $|\alpha| = \sqrt{x^2 + y^2}$.



Once again, let $\alpha, \beta \in \mathbb{C}$ be arbitrary complex numbers. Prove the following facts:

- i. $|\alpha| = \sqrt{\alpha\alpha^*}$.
- ii. $|\alpha\beta| = |\alpha| \cdot |\beta|$. In other words, the magnitude of the product of two complex numbers is the product of their magnitudes.
- iii. $\alpha = 0$ if and only if $|\alpha| = 0$. In other words, the magnitude of a complex number is 0 exactly when the complex number is 0.

3. Linear algebra fundamentals

The following problems are intended to be a review of what you've learned in a previous linear algebra class (Math 18 or similar). At some level, quantum computation consists entirely of multiplying a vector by a matrix. Therefore, these concepts need to become second nature.

A complex matrix $A \in \mathbb{C}^{n \times m}$ is a rectangular array of complex numbers with n rows and m columns. We denote by $A_{i,j}$ the entry at row i and column j .

- (a) **Matrix multiplication:** Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times p}$ be complex matrices. We define the product of these two matrices AB to be the $n \times p$ matrix such that its (i, j) th entry is $(AB)_{i,j} = \sum_{k=1}^m A_{i,k}B_{k,j}$. We can extend matrix multiplication to complex vectors $v \in \mathbb{C}^n$ by thinking of v as an $n \times 1$ matrix.
- i. Prove that matrix multiplication is not commutative. Namely, give an example of 2×2 matrices A and B such that $AB \neq BA$.
 - ii. Prove that matrix multiplication is distributive. That is, you should show that $A(B + C) = AB + AC$ for any matrices $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{m \times p}$, $C \in \mathbb{C}^{m \times p}$.
- (b) **Matrix inverse:** The inverse of a matrix $A \in \mathbb{C}^{n \times n}$ (if it exists) is the unique matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Here, I is the $n \times n$ identity matrix, which has 1's along the diagonal and 0's everywhere else. If matrix A has an inverse, we say that it is *invertible*.
- i. Not all matrices are invertible. For example, the matrix which is 0 everywhere can't have an inverse. Give an example of a 2×2 matrix A such that A is not the all-zeros matrix and yet A still doesn't have an inverse.
 - ii. True/False: If $A^2 = A$, then A is invertible. Give a short explanation.
 - iii. True/False: If $A^2 = I$, then A is invertible. Give a short explanation.
- (c) **Conjugate transpose:** Given any $n \times m$ matrix A , we define its transpose A^T to be the $m \times n$ matrix where the rows and columns have been swapped. Formally, $(A^T)_{i,j} = A_{j,i}$. We define its conjugate A^* to be the $n \times m$ matrix where we take the

complex conjugate of each entry, that is, $(A^*)_{i,j} = (A_{i,j})^*$. We define its conjugate transpose $A^\dagger$ to be the $m \times n$ matrix where we have taken both the transpose and the conjugate of A , that is, $A^\dagger = (A^*)^T = (A^T)^*$. Once again, we can extend these operations to complex vectors $v \in \mathbb{C}^n$ by thinking of v as an $n \times 1$ matrix. That is, v is a column vector, but $v^\dagger$ is a row vector.

- i. Let $A \in \mathbb{C}^{n \times m}$ be a matrix and $v \in \mathbb{C}^n$ and $w \in \mathbb{C}^m$ be vectors. Which of the following are valid linear-algebraic products for all $n, m \geq 1$:

$$vA, \quad v^\dagger A, \quad w^T A, \quad Aw, \quad Av^\dagger?$$

- ii. Prove that $(AB)^\dagger = B^\dagger A^\dagger$ for any matrices $A, B \in \mathbb{C}^{n \times n}$.
- iii. Prove that if $A \in \mathbb{C}^{n \times n}$ is invertible, then $A^\dagger$ is invertible.